Construction of Triply Periodic Minimal Surfaces

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Abstract

We discuss the construction of triply period minimal surfaces. This includes concepts for constructing new examples as well as a discussion of numerical computations based on the new concept of discrete minimal surfaces. As a result we present a wealth of old and new examples and suggest directions for further generalizations.

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1 Introduction

Minimal surfaces offer a great attraction to many disciplines in mathematics and natural science. Some reasons for the common interest lie in the deep problems which open up during closer investigation of their properties, and others in the widespread applications of minimal surfaces in completely different areas of scientific research. In fact, even in pure mathematics, the research on minimal surfaces developed into a number of disciplines which are interested in different aspects of minimal surface theory and use very different analytic tools, for example differential geometry and partial differential equations. This difference in research interest may also be seen for example between differential geometry and natural sciences. In differential geometry there have been a number of great discoveries during the past decade, but since many of them were related to properties such as completeness and finite total curvature they have not found too much interest in natural sciences. And on the other hand problems about minimal surfaces as for example properties of triply periodic minimal surfaces that arose in natural science have not found too much attention in mathematics.

With the present article we would like to contribute to an interdisciplinary discussion. The purpose of the article is to describe triply periodic minimal surfaces and their properties from a mathematical point of view. This includes new principles for construction and tools for numerical experiments based on discrete techniques. Especially, we emphasize that a systematic construction of more and more complicated new examples is possible. This shows that there exists a wealth of triply periodic minimal surfaces, in particular many with the same crystallographic symmetry. We discuss some of the new surfaces we have found and offer directions for further generalization of known examples. Nevertheless, we restrict ourself to surfaces whose fundamental domain for the translational symmetry group is bounded by planar symmetry lines rather than straight lines, because the examples bounded by straight lines are found by crystallographic reasoning rather than analytic existence questions.

In the organization of the paper we have avoided a separate picture section but instead use most surfaces as pictorial explanations of the text. To enhance the use of the paper as a (small) library of triply periodic surfaces it was therefore necessary to include a list-of-figure table at the beginning. Some tables of surfaces show one-parameter deformations between known surfaces, which contain a new surface for an intermediate parameter. In this article we can show only very few intermediate steps - for animations we refer to the recent video on minimal surfaces "Touching Soap Films" [1], [2] which explains minimal surface theory to a popular scientifically interested audience.

The paper starts in section 2 with a review of mathematical properties and continues in section 3 with a discussion of triply periodic minimal surfaces. Section 4 introduces the powerful "conjugate surface construction" which allows to construct fundamental domains for most examples of triply periodic minimal surfaces which are solutions of free boundary problems - this includes even instable ones. For practical purposes a numerical construction of minimal surfaces is a vital question. We include in section 6 an introduction to the concept of discrete minimal surfaces and their properties. For the first time a numerical construction of most known triply periodic minimal surfaces has become an easy task. Specific methods of construction are given in section 5 based on global and local approaches. We close our paper with a prospect on different ways of generalization of triply periodic minimal surfaces whose strong interest may already be seen in natural science.

2 First Properties of Minimal Surfaces

Minimal surfaces have a long history of over 200 years. Research began in the middle of the 18th century when, from the research of the mathematician Lagrange on variational problems, the following question arose: "How does a surface bounded by a given contour look like, when it has smallest surface area?" Lagrange [9] was interested in general variational problems where minimization and maximization of specific properties is analyzed. For mathematicians it was a hard problem to study this new kind of surfaces, called minimal surfaces. The variational problem of minimizing surface area leads to a partial differential equation for the unknown minimal surface, and toolkits for their study were not yet developed in those days. In fact, the first mathematical conjectures about minimal surfaces were derived from the careful observations of soap films by the physicist Plateau [16]. Comprehensive introductions to the historical development of minimal surface theory can be found e.g. in the monographs of Nitsche [14] and Hildebrandt et al. [?].

2.1 Local and Global Definition

When analyzing the properties of surfaces with minimal area it immediately turns out that smaller pieces of the surface also have minimal area with respect to their own smaller boundary. When looking at a very small neighbourhood of an arbitrary point on such a surface it must have minimal area too and therefore look like a "saddle". Around no point the surface can look like a basin or a hill, i.e. it cannot be curved only to one side of the tangent plane because otherwise one could reduce the surface area by cutting the hill off or filling the basin. A surface which does not stay locally on one side of its tangent plane intersects the tangent plane. A horse saddle is the best known example for such a behaviour as indicated in figure 1.

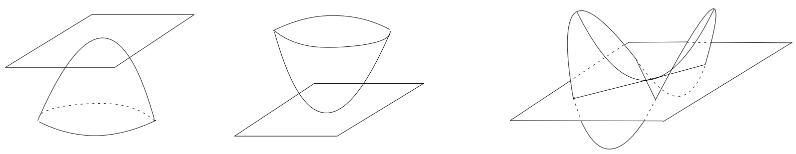

Figure 1: Surfaces are locally like a hill, basin or saddle

But minimal surface saddles are even more special: they look the same from both their sides. From the soap film point of view this is obvious: since the surface tension of a soap film is in equilibrium at every point, the forces pulling to one side must balance the forces which pull to the other side. In differential geometry the term *mean curvature* measures the bending at a point - we just saw, that it must vanish for minimal surfaces.

When looking at a bigger minimal surface, or physically speaking at a bigger soap film, spanned by its boundary it is not true in general, that it has minimal area. There might be other, completely different soap films with the same boundary having less area. Compare figure 2 for an example. But it is true that each of these soap films locally around each point has minimal area and fulfills the balancing condition. Although a mathematically existing minimal surface is defined by fulfilling the local balancing condition everywhere, the surface might not be realizable as a physical soap film, because the soap film might be globally unstable. We repeat that minimal surfaces are defined to have *locally* minimal area, i.e. small portions of a mathematical minimal surface are always realizable as a physical soap film. The definition of vanishing mean curvature makes minimal surfaces independent of the original boundary problem and, in fact, other boundary problems are of similar important interest.

2.2 Boundary Value Problems

The original problem of finding a minimal surface spanned by a given boundary curve is called the *Plateau Problem*, named after the Belgian physicist J.A.F. Plateau [16] who made extensive experimental studies with soap films during the 19th century.

A further boundary problem is the so-called *free boundary problem.* Here a part of the boundary curve is restricted to lie on a given plane instead of being a given curve. The boundary is free to choose its position on the bounding plane. A simple argument shows that the minimal surface must meat the boundary plane at a right angle. Otherwise the surface area could be reduced by a simple modification locally near some boundary point.

Another existence question asks for a surface which is partially bounded by thread. A thread is characterized as having constant length, fixed end points and no resistance to bending.

In differential geometry one is also interested in infinitely large minimal surfaces without boundary. Examples are the simple classical catenoid or the recently discovered Costa surface shown in figure 2. The figures show interesting finite portions of the infinite surfaces.

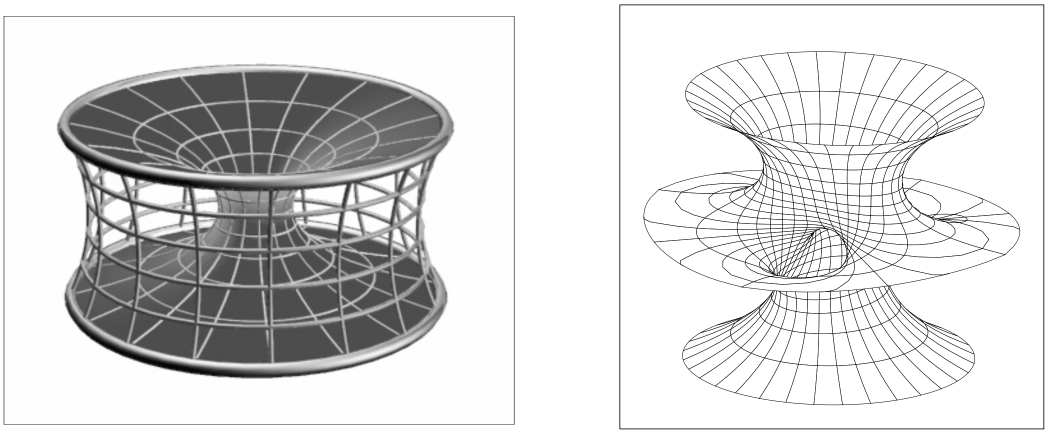

Figure 2: Left: Stable and instable catenoid [cite{Touching}], both surfaces are bounded by the same contour. Right: Costa surface

For the construction of minimal surface there are some remarkable properties of boundary value problems. The most important one is the solvability of the Plateau problem proved by Douglas and Rado:

Theorem 1 *Every simple closed boundary curve spans at least one minimal surface.*

This theorem guarantees a minimal patch for every given surface. But the patch is usually not unique, compare figure 19.

Today we use this result to prove existence also of rather specific problems. The emphasis on existence results in mathematics is not always appreciated outside of our field. One should keep in mind that so-called physical intuition is no longer a reliable guide if the minimal surface under consideration is unstable. Indeed, for the more general boundary problems mentioned above and for nearly all surfaces considered in this article we do not have such a good existence result as theorem 1. And this failure is one of the major obstacles in rigorous constructions of triply periodic minimal surfaces. Later we will describe the *conjugate surface method* which helps in many cases to construct solutions for free boundary problems as they occur in the theory of triply periodic minimal surfaces.

2.3 Symmetry and Embeddedness Properties

Among the major properties used in constructions of new minimal surfaces are symmetry properties:

Lemma 2 (Straight Line) *Every minimal patch bounded by a straight line segment may be extended across the line to a bigger minimal piece: rotate a copy of the original piece by 180° around the straight line. Original piece and copy will not only have the same tangent plane along the line but also all higher derivatives agree, therefore together both patches form a bigger minimal surface.*

To apply this property choose any polygon in $\mathbf{R}^3$ with angles $\frac{\pi}{k}$ and which does not cross itself. The Plateau solution of such a polygon can always be extended to an infinite minimal surface in $\mathbf{R}^3$ without boundary. But in most cases this surface will have too many selfintersections to be interesting.

Lemma 3 (Planar Symmetry) *Every minimal patch which meets a plane orthogonally (e.g. as the result of a free boundary value problem) can be extended by reflection in the plane to a larger patch. As above, in the case of minimal surfaces all derivatives of both patches agree along the plane and therefore form a larger minimal surface.*

Especially the last property is crucial for the conjugate surface construction as we will discuss in greater detail in the section 4. But note, that one may loose stability when doubling the surface: even if the first piece is a stable minimal surface, the doubled one may not.

Lemma 4 (Point Inversion, Normal Rotation) *If a planar symmetry line and a straight line on a minimal surface meet in a point P then inversion $X \to P + (P - X)$ is a symmetry of the minimal surface. If two planar symmetry lines or two straight lines meet under an angle $\frac{\pi}{k}$ in P, then a rotation by $\frac{2\pi}{k}$ around the normal at P is a symmetry of the surface.*

The last symmetries are important because they persist as the minimal surface is deformed through its associate family. In fact, they may occur without the line symmetries.

So far, outside mathematics only pictured minimal surfaces have been accepted as existent. In such cases one can see whether they have selfintersection. In mathematics we look for theorems which prove that there are no selfintersections. The problem of selfintersections for periodic minimal surfaces is often decided by a 2-step argument: at first one proves that the fundamental domain is embedded, and then checks that the continuation with the symmetry (crystallographic) group does not lead to intersecting copies. The arguments for the later step depend on the symmetry group, and to prove embeddedness of the fundamental domain different techniques may be used. For example the following uniqueness theorem and the theorem of Krust [personal communication, see p. 118 in [**?**]] are often applicable:

Lemma 5 *1.)If the boundary of a minimal patch has a 1-1 projection onto a convex planar domain then the patch is the unique minimal surface bounded by its contour and, moreover, it is* embedded *(i.e. without selfintersections).*

2.)(Krust) The associate family of a minimal patch which is a graph over a convex planar domain consists of minimal surfaces which are all graphs and therefore embedded.

3 Triply Periodic Minimal Surfaces

Triply periodic surfaces have by definition translational symmetries in three independent directions. When triply periodic minimal surfaces (in the following abbreviated as TPMS) are considered in the natural sciences it is almost automatically understood that they are without selfintersections. These are also mathematically the most interesting examples because the ones with selfintersections are so abundant that finding them poses no problem.

3.1 Review of Known Examples

At first let us review the set of known surfaces. The most popular examples have the symmetries of a crystallographic group - with a group which is generated by reflections in planes. Such groups have fundamental domains for their translational subgroups which are easy to imagine: convex polyhedra, also called crystallographic cells. The piece of a triply periodic minimal surface inside such a cell is also easy to visualize. It meets the boundary planes of the cell in symmetry lines and it looks like a complicated piece with handles and tunnels, where some of them are opened against the cell's boundary faces.

The most famous of these examples is Schwarz' P-surface [21] shown in figure 16. Schwarz and his students found five triply periodic surfaces. By the way, W. Meeks [11] showed that this surface can deformed to have any translational lattice symmetry. Of course, since only very special lattices allow reflectional symmetries there are usually no symmetries for the deformed minimal surfaces. Meek's examples without symmetry lines have been ignored, probably because of a lack of available pictures.

Around 1970 A. Schoen [20] found many more triply periodic surfaces in crystallographic cells. He made them popular in the natural sciences, but his description as balanced surfaces separating skeletal graphs could not be made into a mathematical existence proof and Schoen remained unknown among mathematicians.

Later Karcher [6] proved existence of Schoen's surfaces using the conjugate surface method. With a refined version of the conjugate surface method Karcher found many more triply periodic examples which are roughly speaking like mixtures of Schoen's examples [7]. We will explain the construction method and a number of examples in later sections. The method is not restricted to the construction of minimal surfaces in euclidean space $\mathbf{R}^3$, e.g. in [8] and in [18] a wealth of periodic minimal surfaces are constructed in $\mathbf{S}^3$ and $\mathbf{H}^3$.

At earlier times pictures of those examples where made from the Weierstraß representation formulas. Such formulas are still unknown for the more complicated examples and deformations, see e.g. the F-Rd surface and the new surfaces in figure tables 18, 20, 12, 16, 15, 13. Today we have discrete techniques available to experiment even with these complicated examples. The wealth of these existing surfaces should be a convincing argument to expect that a crystallographic group can have many minimal surfaces.

In the current paper we restrict ourself to TPMS whose fundamental domain for the translational symmetry group is bounded by planar symmetry lines, because the mathematical considerations for this class of surfaces is mainly related to analytic questions of minimal surfaces. Construction of minimal surfaces with polygonal Plateau contours is mainly a crystallographic problem because theorem 1 assures existence of the minimal patch. This approach was taken by Fischer and Koch [3]. They classified the crystallographic groups which have enough 180°-rotation axes so that Plateau contours made of pieces of these axes are formed. Fischer and Koch used not only boundaries for disc-type fundamental pieces but also for annular fundamental pieces.

Two more surfaces deserve to be specially mentioned. One is the Gyroid of A. Schoen [20]. It is an embedded triply periodic minimal surface and lies in the associate family of Schwarz' P-surface and D-surface. All the symmetry lines of the P-surface (or the D-surface) correspond to curves on the Gyroid which are nearly helices. This explain the name and the difficulty to imagine its shape. Several years ago Lidin [10] informed us that he had found numerically another such surface in the associate family of Schwarz' H-surface. We checked its existence and refer to it as the Lidinoid. It is as intriguing to look at as the Gyroid. See Große-Brauckmann and Wohlgemuth [5] for an embeddedness proof of these surfaces.

3.2 Schwarzian Chains

The very first examples of TPMS where found by Schwarz [21]. Schwarz was working on the general Plateau problem and he followed the approach of constructing a solution to a modified problem: approximate the boundary contour by a polygon and try to find the minimal surface bounded by the polygon. Then increase the discretization of the polygon and hope to find a converging sequence. This method did not work in the end, but during that research Schwarz found a number of interesting other results.

In 1816 Gergonne posed the following problem [4]: is it possible to bisect a cube in such a way, that the intersection surface is bounded by the inverse diagonals of two opposite faces of the cube, and that the intersection surface has smallest area? The solution of the problem was awarded with a price, which was received 20 years later by Scherk [19] for finding similar surfaces - but the Gergonne problem stayed open.

Schwarz used complex analysis and the Weierstraß formula for constructing new surfaces. One of the spectacular results of his methods was the solution of the Gergonne problem in 1865. Schwarz was able to find the Weierstraß functions for many so-called Schwarzian chains. These are mixed boundary contours consisting of straight arcs and planar symmetry lines, i.e. free boundary problems. The contour of the Gergonne problem can be seen as a Schwarzian chain, compare figure 10.

4 Conjugate Surface Method

Over the last decade the conjugate surface method has been established as one of the most powerful techniques to construct minimal surfaces with a proposed shape in mind. For periodic surfaces the method is very easy to explain and we will do it in this chapter. We will also mention the difficult aspects when constructing more complicated examples and we will explain a numerical approach applicable even where theoretical techniques fail up to now.

4.1 Associate Family of a Simple Example

Among the fundamental observations in minimal surface theory in the last century was that every minimal surface comes in a family of minimal surfaces, the so-called associate family or Bonnet family. Before going into details, the simplest and most popular example is the associate family in which the catenoid deforms into the helicoid: The catenoid is given by

$$C(u,v) = \begin{pmatrix} \cos v \cosh u \\ \sin v \cosh u \\ u \end{pmatrix}$$

and the helicoid by

$$H(u,v) = \begin{pmatrix} \sin v \sinh u \\ -\cos v \sinh u \\ v \end{pmatrix}.$$

With the following weighted sum we obtain the associate family $F^\varphi(u,v)$ of both minimal surfaces:

$$F^\varphi(u,v) = \cos\varphi \cdot C(u,v) + \sin\varphi \cdot H(u,v).$$

The parameter $\varphi \in [0, 2\pi]$ is the family parameter. For $\varphi = \frac{\pi}{2}$ the surface is called the *conjugate* of the surface with $\varphi = 0$, and $\varphi = \pi$ leads to a point mirror image. The helicoid is called the *conjugate surface* of the catenoid, and in general each pair of surfaces F^φ and $F^{\varphi+\frac{\pi}{2}}$ are conjugate to each other.

Theorem 6 *The following properties of conjugate surfaces and the associate family are easily verified by direct computation, compare also figure 3 for a pictorial explanation:*

I *The surface normals at points corresponding to an arbitrary point (u_0, v_0) in the domain are identical, i.e. $N_{F^\varphi}(u_0,v_0) = N_C(u_0,v_0) = N_H(u_0,v_0)$,*

II *The partial derivatives fulfill the following correspondence:*

$$\begin{array}{rcl} F^\varphi_u(u_0,v_0) & = & \cos\varphi \cdot C_u(u_0,v_0) - \sin\varphi \cdot C_v(u_0,v_0) \\ F^\varphi_v(u_0,v_0) & = & \sin\varphi \cdot C_u(u_0,v_0) + \cos\varphi \cdot C_v(u_0,v_0) \end{array},$$

in particular, the partials of catenoid and helicoid satisfy the Cauchy-Riemann equations:

$$\begin{array}{rcl} C_u(u_0,v_0) & = & H_v(u_0,v_0) \\ C_v(u_0,v_0) & = & -H_u(u_0,v_0) \end{array}.$$

III *If a minimal patch is bounded by a straight line, then its conjugate patch is bounded by a planar symmetry line and vice versa. This can be seen in the catenoid-helicoid examples, where planar meridians of the catenoid correspond to the straight lines of the helicoid.*

IV *Since at every point the length and the angle between the partial derivatives are identical for the surface and its conjugate (i.e. both surfaces are isometric) we have as a result, that the angles at corresponding boundary vertices of surface and conjugate surface are identical.*

The last two properties are most important for the later conjugate surface method.

A more suitable notation for the relation above is given in terms of complex analysis as it is done in mathematics. (Of course, this notation is not essential for our subsequent algorithms and can be overlooked by non-experts.) If we use complex notation then $C + i \cdot H$ is a complex curve in $\mathbf{C}^3$ and with the complex coordinate $z = u + iv$ we have

$$F^\varphi(z) = \mathrm{Re}(e^{-i\varphi} \cdot (C(z) + i \cdot H(z)) = \mathrm{Re}(e^{-i\varphi} \cdot \begin{pmatrix} \cosh z \\ -i \sinh z \\ z \end{pmatrix}).$$

In a complex analytical description of minimal surfaces the associate family is an easy concept. It is a basic fact in complex analysis that every harmonic map is the real part of a complex holomorphic map. The three coordinate functions of a minimal surface in euclidean space $\mathbf{R}^3$ are harmonic maps. Therefore, there exist three other harmonic maps which define together with the original coordinate functions three holomorphic functions, or a single complex vector-valued function. The real part of this function is the original minimal surface, the imaginary part is the conjugate surface, and projections in-between define surfaces of the associate family.

4.2 The Construction Method

The conjugate surface method has been very successful in solving free boundary problems. As an example, consider the boundary problem for the fundamental patch of Neovius [12] surface in figure 3 or with an additional handle as shown in figure 6. If a minimal patch exists with free boundaries at some faces of the tetrahedron as required by the surface, then the conjugate patch in the associate family exists and must be bounded by straight lines. This observation can be reversed to a construction principle: it is sufficient to prove existence of a minimal patch in a corresponding boundary polygon of straight segments (the existence is proved by the solution of the general Plateau Problem), then one conjugates it and has a solution of the required free boundary contour. The only remaining problem is to find for a given free boundary contour the corresponding conjugate polygon of straight segments.

To solve this problem two of the properties of minimal surfaces listed above will help us:

- We know that both patches are isometric, therefore each vertex angle between two adjacent straight segments is identical to the dihedral angle spanned by the two corresponding planes in the free boundary problem.
- The normal vectors at corresponding vertices are identical. The normal vectors at vertices of the free boundary problem are just the direction of the intersection line of two adjacent boundary planes, and can therefore be read from the free boundary specification.

If the patch has four boundary curves, these two properties uniquely determine the corresponding polygonal contour for a free boundary problem up to scaling. We can now specify detailed instructions for the conjugate surface method:

Theorem 7 (Construction of the Polygon) *Given a free boundary problem, i.e. a set of planes which shall be met orthogonally by the searched minimal surface. If this is a well-posed problem we can construct the conjugate contour consisting of straight lines:*

I *Read the vertex angles and vertex normals from the arrangement of boundary planes.*

II *Start to generate the conjugate polygonal contour at an arbitrary vertex. Leave the vertex a certain distance along the straight segment, which is orthogonally to the corresponding plane.*

III *At the end of the current straight segment repeat the process, but the direction of the next segment is now determined by the vertex normal and the known angle between both straight segments. Repeating this process usually leads to a non-closed polygonal contour.*

IV *It remains to adjust the edge lengths such that the contour closes. If the contour consists of four vertices, the edge lengths are uniquely determined up to scaling of the whole contour. For N vertices in the contour one has in general $N - 4$ edge lengths to choose, the others are determined by the closing condition.*

V *Given the polygonal contour the construction is finished: its conjugated Plateau solution gives a solution for the original free boundary problem.*

A difficulty comes into the game if two boundary curves in the original free boundary problem span the same plane. Since only normal vectors in the above algorithm are used, the construction cannot distinguish between two parallel planes. Therefore the resulting patch may have all boundary symmetries as required but the two planes are not identical. Compare figure 6 for a pictorial description of the problem. The problem of having two

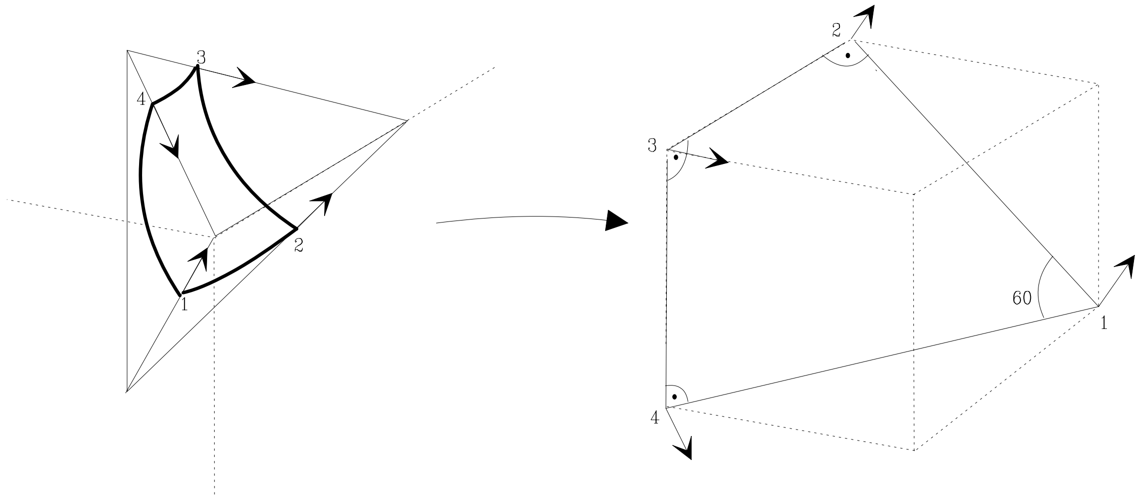

Figure 3: Construction of the conjugate polygonal contour: read vertex angles and vertex normals from the searched free boundary problem (left) and construct a polygon (right) with the same data. A contour with N edges will lead to N-4 unknown edge lengths, i.e. period problems

parallel planes is usually called *Period Problem*, because two parallel planes result in an additional unwanted translational period. For more complicated boundary problems the period problem is the major obstacle in existence proofs with the conjugate surface method.

4.3 The Period Problem

The so-called period problem is the major obstacle in a successful application of the conjugate surface method. As an introductory example we will try to insert an additional handle in all four horizontal handles of Schwarz' P-surface, compare figure 11 and 4.

This was first carried out by Karcher [6]. We already know the corresponding contour of the fundamental domain of Schwarz' P-surface. Applying the conjugate surface method results at the end in the following modification of the polygonal contour: replace the vertex which corresponds to the place where the handle will be inserted with a straight segment orthogonally to the new symmetry plane of the handle as shown in 5.

The length of the additional straight segment is equal to the perimeter of the handle segment. Therefore, a small segment will result in a small handle and the resulting surface is close to Schwarz' original P-surface. After reflection the two handles emanating from top and bottom will not close up. Adjusting the perimeter is necessary so that the rim of the top half-handle coincides with the rim of the bottom half-handle. Otherwise one would be left with two additional symmetry planes and the resulting TPMS would not be embedded.

There is no general theorem to solve the period problem. And in fact, in many cases there exists no solution. Since a rigorous solution of the period problem is usually a

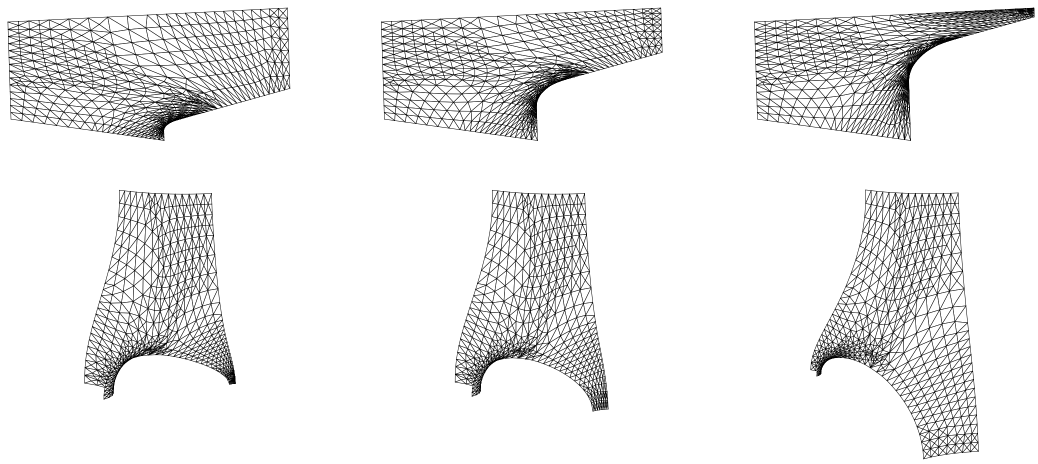

Figure 4: Fundamental domains for inclusion of handles into Schwarz' P-surface. First line: Plateau surfaces with straight boundary segments. Second line: the desired fundamental pieces conjugate to the above Plateau solutions.

difficult task one must rely for practical purposes and experimental studies on numerical computations. For triply periodic surfaces one often does not know the Weierstraß representation functions, but one can use the recently developed concept of discrete minimal surfaces.

5 Concept of Handle Insertion

A very successful method in the construction of minimal surfaces is the *concept of handle insertion*. Many new examples and modifications of known surfaces where constructed using this method. Our pictures only show a small sample from this rich collection. In the following we will explain the concept in detail and thereby give a step-to-step application of the conjugate surface method in case of a one-parameter problem.

Let us start with some heuristics. Consider a soap film with a large flat region, for example as they occur on minimal surfaces where two planar symmetry lines meet at an angle of $\frac{\pi}{k}, k > 3$. A relatively large neighbourhood around such a point is a stable minimal patch inside its boundary and one can expect to some extent that a tiny modification of the surface at such a point should not disturb the whole minimal surface too much. For example, let us put a small ring onto the soap film at such a point and cautiously pull it off the surface. As long as the film does not burst we see a small handle developing. Compare for example figure 14 where at the center of every face of the I-Wp surface small handles develop to become the O,C-TO surface. Surely, the heuristic example is not quite correct, since modification of existing minimal surfaces immediately modifies the surface even at points far away.

But as the example of the O,C-TO surface shows one can expect to control the construction in some cases. In the following we will assume that the boundary of a new handle is a planar symmetry line since we want to construct triply period minimal surfaces. Also, it will turn out that for inserting new handles it is not necessary to restrict to flat areas.

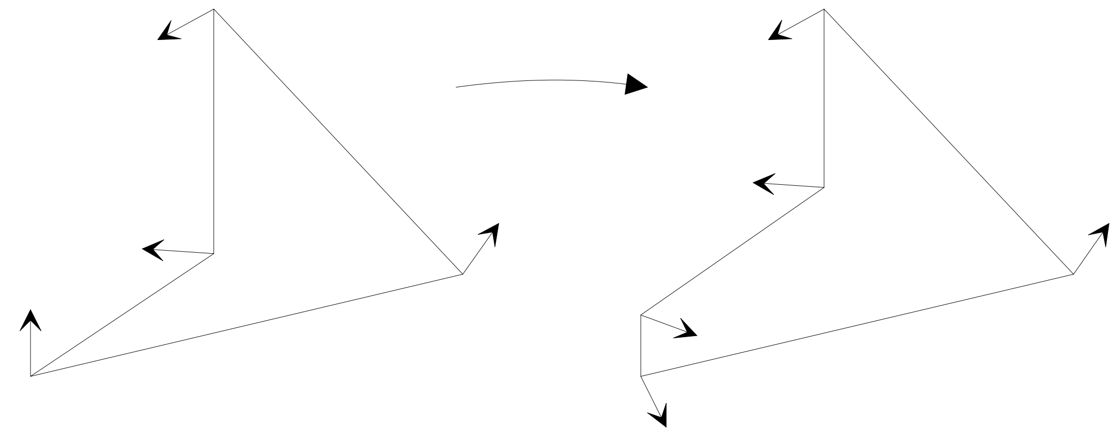

Figure 5: Usually one inserts an additional handle at the vertices of the fundamental domain and chooses the axis of the handle to be parallel to the original surface normal. At the conjugate contour this is done by inserting an additional edge parallel to the surface normal. Compare with next figure.

Let us take the fundamental patch of an existing minimal surface, e.g. the surface of Neovius. This patch is bounded by the faces of a tetrahedron. We specify a modified free boundary problem by searching for a minimal patch with an additional boundary curve on the tetrahedron, which will later result in a new handle at the center of each cubical face of Neovius' surface. The original free boundary solution for the fundamental patch of Neovius' surface was already an unstable patch, and this is even more true for the more complicated patch we have in mind now. Therefore neither computing the patch numerically with a direct minimization algorithm will work without further additional tricks nor will easy theoretical concepts directly prove existence of such a patch.

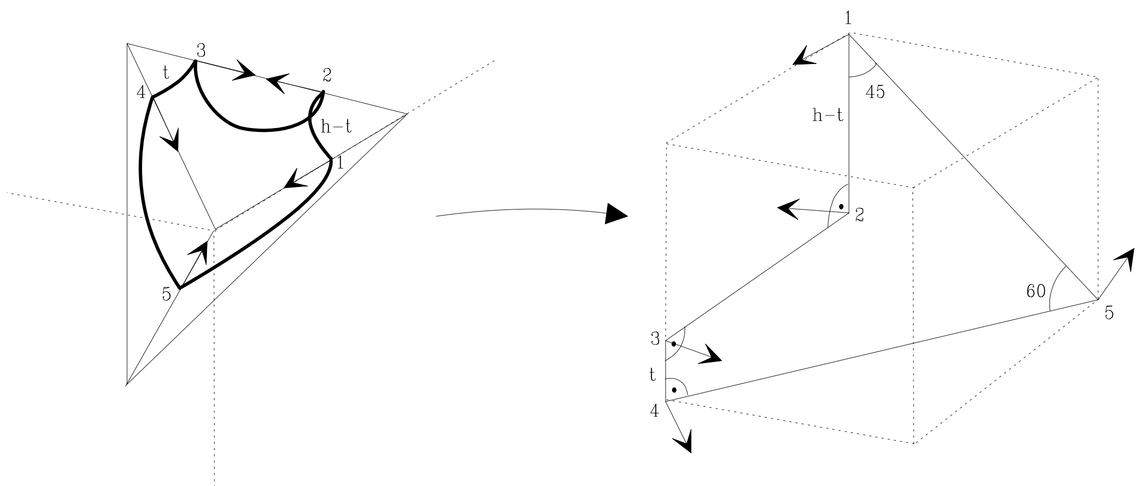

Figure 6: A standard period problem: the new handle inserted at vertex 2 in figure [fig.conj] adds a new symmetry arc between vertex 1 and 2 which must be equal to the existing symmetry arc between vertex 3 and 4. The parameter t varies between 0 and h, thereby making the handle smaller resp. bigger.

At this point the conjugate surface method can show its full advantage. Instead of constructing the patch directly we construct its conjugate patch. The conjugate patch is bounded by straight lines as we have seen in chapter 4 and we can compute its polygonal contour from the information we get from the way in which the searched patch must lie inside the tetrahedron, compare figure 6:

- Patch and polygonal contour must have the same vertex angles, therefore we read from the tetrahedron, that the polygonal contour must have vertex angles $45°, 90°, 90°, 90°$ and $60°$ at corresponding vertices.
  - In addition we know that the normal vectors at corresponding vertices are identical and that every polygonal segment is orthogonal to the plane of its corresponding planar arc.
  - Let us draw the first vertex of the polygonal contour and its normal. Then we emanate in one of the two directions orthogonally to the plane of the first planar arc, in figure 5 we have chosen to go down. We have to assume an arbitrary length for the first arc, and stop somewhere to define the second vertex.
  - Again, let us mark at that point the second vertex and draw its normal vector. Now the direction in which we must leave to the third vertex is uniquely given by the vertex angle and the normal vector.
  - At every step we assume an arbitrary edge length and continue with this procedure until we reach the last vertex, number five in our example. The condition that the polygonal contour must close uniquely determines the pairwise ratios of the second, fourth and fifth edge length. The first and third edge length may instead take on two arbitrary values which must sum up to the total height h of the cube. h may be normalized to 1, other values will only scale the minimal surface.

Let us define the length of edge three as our parameter t, then length of edge one is determined as $h - t$. That means, we have a 1-parameter family of possible contours. Each contour will lead to a minimal patch with required symmetries, but only one will lie in the tetrahedron (in fact, uniqueness is not guaranteed). We shall now go on to construct the minimal surface patch.

- Every polygonal contour of the 1-parameter family bounds a minimal surface, its Plateau solution.
  - Conjugating the 1-parameter family leads to a family of surfaces which all have the required symmetries.
  - If for some parameter values the period is negative and for others positive then there exists an intermediate parameter value t_0 with vanishing period. This can be seen if one can show that the two limit surfaces corresponding to parameter values $t = 0$ and $t = 1$ exist and have opposite sign in their limit period: for $t = 0$ we have Neovius' surface, i.e. the inserted handle is too small (period is negative), and for $t = 1$ we have Schwarz' surface whose handle is far too big (period is positive).
  - The surface corresponding to t_0 is the final fundamental patch for Neovius' surface with additional Schwarz handles whose existence we have just proved.

The period can be explicitly measured, it is in our case the distance between the symmetry plane of the top of the inserted handle and the parallel symmetry plane of the Neovius handle. If both symmetry planes are not identical then reflection in both planes will generate an additional translation orthogonal to the planes. Therefore the resulting surface will have selfintersections as long as the period does not vanish.

In this existence proof we used continuous dependence of Plateau solutions on their boundary contour, i.e. the period depends continuously on the polygonal contour. In full generality this is false, but for our surfaces the family of contours can be parallel projected to the boundary of a fixed convex domain. In such cases a result of Nitsche assures continuity [13].

5.1 Other Handle Types

There are different ways to look at handles. Their name suggests that handles are small but we have seen already that they may become big and dominate the surface. Therefore there is often some ambiguity when refering to handles. But from the constructive point of view it is clear what a handle means.

Classically, a handle is a cylindrical connection between two objects. If one considers two adjacent cells of Schwarz P-surface, then both are connected by such a classical handle. The way we construct handles, the handle is symmetric w.r.t. its waist. We call the handle *Schwarz handle.*

Increasing the symmetry of the classical handle leads to the Neovius handle type: It consists of half handles centered at a point and forming a regular star. The handles of Neovius surface form such a configuration around the edges of the bounding cube.

Similarly one can increase the symmetry to three dimensions and obtain handles which point in each direction of all faces of a Platonic solid. They are called I-Wp handles since the fundamental cell of the I-Wp surface of A. Schoen forms such a handle with octahedral symmetry (the I-Wp surface is delicate since every building block of the handle has a rotational symmetry exchanging its ends, therefore one can become confused, because the fundamental cell of the I-Wp surface is bounded by a cube. But one should observe how the handle parts of the I-Wp surface are connected in the center of the cube). It is best to name these kind of handles by the Platonic solid they correspond to.

6 Discrete Minimal Surfaces and Numerics

Numerical computations of minimal surfaces may be done with a number of different techniques. Classically, most examples where computed using the following two methods: The first method assumes knowledge of the Weierstraß integration formulas and does a numerical integration. For complicated surfaces this is a non-trivial task, but the method has never failed in such cases up to now. The second method relies on the finite element theory and tries to solve the underlying partial differential equations with numerical techniques. Both methods have drawbacks when applied to the computation of triply periodic minimal surfaces: the first method needs the Weierstraß formula functions which are difficult to derive for more complicated examples, and the second method relies on energy minimization techniques, which will also have difficulties since fundamental domains of more complicated examples are usually not stable minima for a known energy functional. They are only critical points, and therefore direct minimization will usually miss these examples. To deal with this problem one needs to construct additional restrictions depending on each investigated example.

In this chapter we will introduce a different method based on discrete techniques. *Discrete* means, that surfaces are not considered as smooth objects but for example as a combinatorial complex of triangles. A similar approach is also made in finite element theory but there one always has a smooth limit surface in mind which would be obtained when the discretization level approaches zero. The instability of the solution of the free boundary problem is avoided because the discrete approach can handle the conjugate surface method. The derivation of Weierstraß data is avoided (and often impossible), since the discrete minimal surface patches will be defined by their polygonal contours.

Let us start with a definition of a discrete minimal surface. For simplicity, we restrict ourself here to triangulated discrete surfaces, but the reader should keep in mind that already during the conjugate surface construction discrete surfaces with other combinatorial structure occur:

Definition 1 *A* discrete surface *is a collection of triangles which have the structure of a topological simplicial complex, i.e. any two triangles are either disjoint or have a single edge in common or a single point.*

This definition covers ordinary triangulated surfaces. But it also includes, for example, edges where several triangles meet as it occurs in experiments with soap foam. This is out of the scope of the present paper.

Let us now refine the above definition to the case of area minimizing discrete surfaces. The area of discrete surfaces is defined to be the sum of the areas of each individual triangle. But as in the case of smooth minimal surfaces we make the definition more general and include those surfaces which minimize area only locally (i.e. which may globally not be area minimizer):

Definition 2 *A* discrete minimal surface *is a discrete surface with the property that no single vertex of the triangulation can be moved to decrease its area, i.e. the surface is a critical point for the discrete area functional.*

It turns out that the minimality condition at every vertex can be described by an explicit formula in terms of geometric scalars. Consider figure 7 where the local neighbourhood of a point on a discrete surface is shown.

Lemma 8 (Balancing Condition) *The following formula describes a balancing condition every discrete minimal surface fulfills. The condition should be seen in analogy to the surface tension of soap films which balances at every point. Mathematically the relation describes the vanishing of the gradient of the area at a point p. It must be fulfilled at every point p if the surface is a discrete minimal surface:*

$$\frac{\partial}{\partial p} Area(triangulation) = \sum_{i=1}^{\#neighbours\ of\ p} (\cot\alpha_i + \cot\beta_i)(p - q_i) = 0 \tag{1}$$

The formula can also be interpreted as a weighted sum of the edges $p - q_i$ emanating at p. The weight factors $\cot\alpha_i + \cot\beta_i$ are computed using the cotangent of the two angles which lie in the two adjacent triangles opposite to the edge. Figure 7 explains it

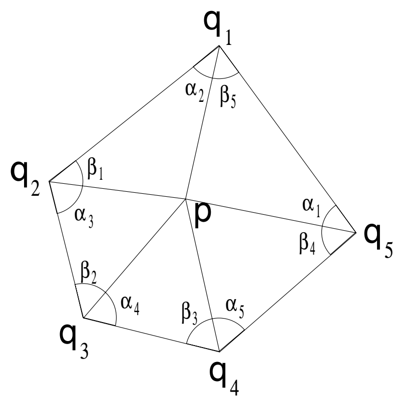

Figure 7: Neighbourhood around a point on a discrete surface

pictorially. This interpretation explains the analogy to the tension of a soap film which also balances at every point. The analogy to the smooth situation can be driven further to define the vector-valued sum as the equivalent of the smooth laplacian, i.e. as the discrete laplacian. Similar to the smooth case the discrete laplacian must vanish for discrete minimal surfaces.

A remarkable and unexpected fact is the simplicity of the minimality condition. Of course, the same formula must come out when using finite element theory, but there one would not see the influence of the geometric terms like angles and edges in such a clear way. It would be hidden behind integrals of finite element basis functions. Also we have here the opportunity to go further and to conjugate the discrete minimal surface. This is not a trivial task and it was done in the paper of Pinkall and Polthier [15]. The fact, that *conjugation* can be defined for discrete minimal surfaces in such a way that similar properties as in the smooth case remain true should be considered as a substantial success of the discrete concept and the definition of discrete surfaces. We do not go into the details of the conjugation algorithm in this paper and refer to the reference above.

6.1 Solving Stable and Instable Problems

In the numerical practice we do not solve the above equilibrium condition 1 directly but use instead an iteration process based on a different energy functional, as described in detail in [15]. At the boundary the gradients are restricted in such a way that the boundary conditions are always fulfilled during minimization. This allows to apply the minimization algorithm also e.g. to free boundary problems. In such a case the gradient is projected onto the plane of the boundary, thereby restricting motion of the boundary points in the required way. But solving free boundary problems with a direct minimization approach only succeeds if the resulting minimal patch will be stable. And for most of the more complicated surfaces this is not the case: their fundamental patch is an instable minimal patch, i.e. when we apply a numerical method to experimentally find the minimal patch, then it will be minimized further and usually degenerate to e.g. an edge of the bounding polyhedron, compare figure 8. In such instable cases the numerical conjugate surface method is still applicable if the conjugate contours have stable Plateau solutions.

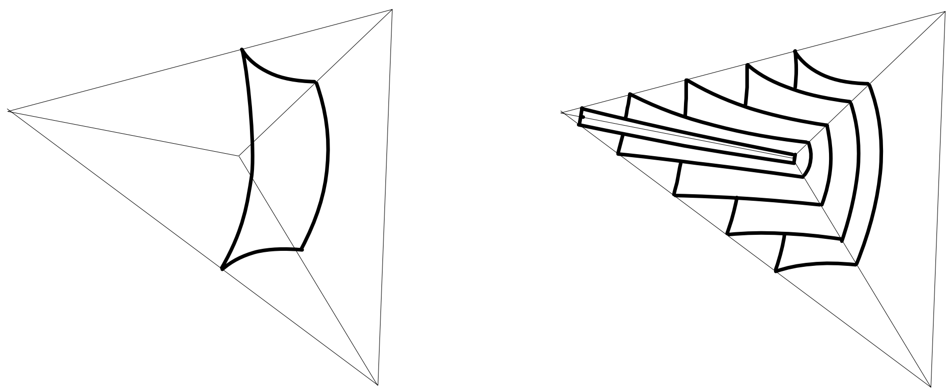

Figure 8: The fundamental pieces of Schwarz-P, Neovius and I-Wp surfaces considered as free boundary value problems in a tetrahedron are instable surfaces. Minimizing their energy leads to degeneration as shown in the right figure

6.2 Numerics of the Period Problem

When working with the conjugate surface method in most cases the conjugate contour is only known up to a number of free parameters. These parameters are the lengths of some edges of the polygonal boundary. Wrong values of the edge lengths will result in so-called periods in the final minimal surface, as it is explained in detail in section 5. For simplicity consider the case of only one free parameter, i.e. one period which must be controlled. If one chooses the corresponding edge length in the polygonal contour too small, the period will be, let's say, negative and if it is too large the period will be positive. A zero-length period is known to exist in-between by continuity arguments. Numerically we choose a set of different edge lengths which are in some way distributed among the possible edge lengths and compute the surface and its conjugate for all these lengths. Then we apply an interpolation technique between the resulting surfaces and obtain for some parameter value a vanishing period.

On first sight this sounds quite trivial but in practice there are concepts needed to cope with the occurring problems. For example, one usually applies adaptive refinement of the triangulations at regions with high curvature as it can be seen in the family of the O,C-TO surface in figure 14. Such situations already require techniques to interpolate among a sequence of surfaces with different triangulations.

6.3 Numerical Sample Session

Finally, let us briefly list the necessary steps to compute a minimal surface with our program:

- Specify major vertices of the boundary contour (i.e. four vertices for a quadrilateral) and the type of boundary curves in a definition file.
- Load the definition file into the program and invoke the automatic surface builder which generates a triangulated surface inside the boundary contour. The number of triangle can be chosen interactively by adaptive refinement depending on the surface curvature.
- Invoke the minimization algorithm to compute the corresponding discrete minimal surface.

If the contour specification belonged to the conjugate minimal patch one must now continue with:

- Apply the conjugation algorithm.

In the original boundary specification one may already mark boundary vertices so that they may depend on further parameters like for example edge lengths. This allows specification of contours which depend on one (or more) parameter, i.e. a family of boundary curves. In the following the program automatically generates initial contours for several parameter values and then applies all operations to the family as a whole, i.e. the user works as above - the only difference is that minimization takes more time since finitely many surfaces of the family must be minimized. At the end the user must interactively check where the period is closed.

- Check vanishing periods in surface family.

Additional operations might be invoked in case of more complicated examples. This includes for example adaptive refinement, reflection of the resulting minimal patch to a larger surface, or computation of the crystallographic cell.

References

[1] A. Arnez, K. Polthier, M. Steffens, and C. Teitzel. Palast der Seifenhäute. Bild der Wissenschaft, 1995. Video.

[2] A. Arnez, K. Polthier, M. Steffens, and C. Teitzel. Touching Soap Films (video). 1995.

[3] W. Fischer and E. Koch. On 3-periodic minimal surfaces. *Zeitschrift für Kristallographie*, 179:31–52, 1987.

[4] J. Gergonne. Questions proposées/résolues. *Ann. Math. Pure Appl.*, 7:68, 99–100, 143–147, 156, 1816.

[5] K. Große-Brauckmann and M. Wohlgemuth. The gyroid is embedded and has constant mean curvature compagnions. *Calculus of Variations*, to appear.

[6] H. Karcher. The triply periodic minimal surfaces of A. Schoen and their constant mean curvature compagnions. *Man. Math.*, 64:291–357, 1989.

[7] H. Karcher. Construction of higher genus minimal surfaces. In *Geometry and Topology of Submanifolds III*, pages 174–191. World Science Pub., 1990.

[8] H. Karcher, U. Pinkall, and I. Sterling. New minimal surfaces in S^3. *J. Diff. Geom.*, 28:169–185, 1988.

[9] J. L. Lagrange. *Oeuvres.* Gauthier-Villars, Paris, 1867.

[10] S. Lidin and S. Larsson. Bonnet transformation of infinite periodic minimal surfaces with hexagonal symmetry. *J. Chem. Soc. Faraday Trans.*, 86 (5):769–775, 1990.

[11] W. H. Meeks. The theory of triply-periodic minimal surfaces. *Indiana University Math. Journal*, 39 (3):877–936, 1990.

[12] E. R. Neovius. *Bestimmung zweier specieller periodischer Minimalflächen, auf welchen unendlich viele gerade Linien und unendlich viele ebene geodätische Linien liegen.* J.C. Frenckell & Sohn, Helsingsfors, 1883.

[13] J. C. C. Nitsche. Über ein verallgemeinertes Dirichletsches Problem für die Minimalflächengleichung und hebbare Unstetigkeiten ihrer Lösungen. *Math. Ann.*, 158:203–214, 1965.

[14] J. C. C. Nitsche. *Lectures on Minimal Surfaces.* Cambridge Univ. Press, 1989.

[15] U. Pinkall and K. Polthier. Computing discrete minimal surfaces and their conjugates. *Experim. Math.*, 2(1):15–36, 1993.

[16] J. A. F. Plateau. *Statique expérimentale et téoretique des liquides soumis aux seule forces moléculaires*, volume I, II. Gauthier-Villars, Paris, 1873.

[17] K. Polthier. Geometric data for triply periodic minimal surfaces in spaces of constant curvature. In P. Concus, R. Finn, and D. Hoffman, editors, *Geometric Analysis and Computer Graphics*, MSRI-Book Series, pages 139–145. Springer-Verlag, 1991.

[18] K. Polthier. New periodic minimal surfaces in h^3. In G. Dziuk, G. Huisken, and J. Hutchinson, editors, *Theoretical and Numerical Aspects of Geometric Variational Problems*, volume 26, pages 201–210. CMA Canberra, 1991.

[19] H. F. Scherk. Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen. *J. Reine Angew. Math.*, pages 185–208, 1835.

[20] A. Schoen. Infinite periodic minimal surfaces without selfintersections. Technical Report Technical Note D-5541, NASA, Cambridge, Mass., 1970.

[21] H. A. Schwarz. *Gesammelte Mathematische Abhandlungen.* Springer, Berlin, 1890.